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A novel discrete grey Riccati model and its application

Liang Zeng
Guangdong Polytechnic College, Zhaoqing, China

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Abstract

Purpose – To develop the theory and application of the grey prediction model, this investigation constructs a novel discrete grey Riccati model termed DGRM(1,1).

Design/methodology/approach – By examining a special kind of Riccati difference equation and the structure of the conventional discrete grey model (DGM), we advance a novel DGRM, and the model's prediction effect is evaluated by two numerical examples and an application case and compared with that of other conventional grey models.

Findings – The average relative simulation error of DGRM(1,1) does not change if the model is built after the original sequence has been transformed by a multiplier, and the new model is suitable to predict monotonically increasing, monotonically decreasing and unimodal sequences.

Practical implications – DGRM(1,1) is utilized to forecast the development cost of a small plane owned by the Aviation Industry Corporation of China (AVIC) with an original data sequence from 2006 to 2013. The outcomes indicate that DGRM(1,1) exhibits high precision and potential in development cost prediction.

Originality/value – Combining the Riccati difference equation with the conventional DGM, the author advances a new grey model that is suitable to predict three kinds of data series with different changing trends.

Keywords Grey system, Riccati equation, Discrete grey model, GM(1,1), Grey Riccati model

Paper type Research paper

1. Introduction

Grey system theory (Deng, 1982) was proposed in 1982. It is mainly used to address uncertain system problems with small samples and little information. From the current research, scholars pay more attention to the grey prediction methods in this theory, and they look upon a system as a continuous function, which changes with time. It does not need many time-series data to obtain a good prediction effect and high precision when modeling. The first-order grey model (GM) with one variable (GM(1,1)), as the basic model of grey prediction theory, has been applied in various fields. The accumulative generation operator is an essential element of GM(1,1) and the other grey prediction models, and it can be considered a special technique of data transformation (Wei *et al.*, 2020). Through accumulation of the outputs of an observed system, a chaotic nonnegative time series can show an approximate exponential growth trend so that the features and integral laws hidden in the series can be fully mined by establishing a first-order differential equation (also called the whitening differential equation) (Wu *et al.*, 2013a). By solving this differential equation, a time-response function for prediction can be obtained. From it, the simulation and prediction value of the first-order accumulation sequence can be obtained. Finally, the prediction results of the original sequence can be calculated by a recursive reduction formula. The identification process of model parameters is to discretize the first-order differential equation to obtain a difference equation and then employ the least square method to solve it. GM(1,1) exhibits a good simulation effect on sequences that approximately obey the homogeneous exponent law, but it is difficult to achieve the ideal effect on sequences with other characteristics. Therefore, to adapt to the changing characteristics of different sequences to achieve high prediction accuracy, researchers have optimized or

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extended the GM(1,1) model and obtained a series of research results. For example, to make most of the new information of the system, Wu *et al.* (2013b) extended the first-order accumulation to the fractional-order accumulation and constructed the GM(1,1) model based on fractional-order accumulation. Ma *et al.* (2019) used the Simpson formula to reconstruct the background value and presented an improved GM(1,1) model. Cui *et al.* (2013) replaced the constant term of the whitening differential equation of GM(1,1) with a linear time function and developed a new model (NGM(1,1)), which was fit for the prediction of a time series with a non-homogeneous exponential law. Qian *et al.* (2012) constructed a GM involving a time-power term (GM(1,1, t^{α})), in which the variability of α allows the model to have strong adaptability. Chen (2008) replaced the traditional grey differential equation with the Bernoulli equation and proposed a nonlinear grey Bernoulli model (NGBM(1,1)). Wang *et al.* (2018) combined the accumulation generation operator with seasonal factors and developed a seasonal GM(1,1) model, which can effectively identify the seasonal fluctuation characteristics contained in a time series. Luo and Wei (2017) substituted the constant term of the whitening differential equation of GM(1,1) for the polynomial function and put forward a grey polynomial model (GPM(1,1,N)), which could adapt to a time series with various changing characteristics.

Although the above models broaden the application scopes of GMs, because each time-response function for prediction is obtained by solving the whitening differential equation and the parameters of the model are obtained by solving the difference equation after discretization of the differential equation, discretization errors inevitably exist. To eliminate these errors and improve the modeling accuracy of the GM, Xie and Liu (2005) proposed a discrete GM(1,1) model (DGM(1,1)), whose basic form is a first-order difference equation, and the identification values of the model parameters and the time-response formula are all based on this equation. Therefore, in most cases, DGM(1,1) can better describe the system's development trend than GM(1,1). To further improve the prediction performance, some other DGMs are proposed based on DGM(1,1). For example, Xie *et al.* (2013) proposed a non-homogeneous DGM (NDGM(1,1)) for a non-homogeneous exponential time series by replacing the constant term in DGM(1,1) with a linear time function. Wu *et al.* (2014a, b) replaced the first-order accumulation operator with the fractional-order accumulation operator and constructed the DGM and the non-homogeneous DGM based on fractional order accumulation, which could effectively improve the prediction accuracy of the original models. Yang and Zhao (2015) developed a discrete grey power model by replacing the constant term in DGM(1,1) with a power function and extended the new model to a fractional-order discrete grey power model by introducing the fractional order accumulation operator. Wei *et al.* (2019) substituted the constant term in DGM(1,1) for the N-order polynomial function and established a discrete grey polynomial model (DGPM(1,1,N)), which could adaptively select the optimal prediction model according to the sequence characteristics. Ma and Liu (2016) presented a discrete multivariable GM (RDGM(1,n)) on the basis of the multivariable GM (GMC(1,n)) (Tien, 2005) and showed that RDGM(1,n) has better prediction performance than GMC(1,n) in most cases. The construction of the above DGMs greatly promotes the development of grey prediction theory. However, in practical applications, the prediction accuracy of the existing models for some sequences is not ideal and still needs to be improved. Therefore, it is necessary to study grey prediction methods more widely and develop more new prediction models to adapt to complex sequences.

The Riccati difference equation is an important class of nonlinear difference equations. It is the basis of studying and solving discrete-time optimal control and filtering problems and has been used widely in some engineering fields, such as hydrodynamics and elastic vibration theory (Zhang and Dower, 2015; Wang *et al.*, 2019; Chen and Shon, 2019). To further develop the theory of the grey prediction model, by using the construction method of the DGM and introducing a special kind of Riccati difference equation to replace the traditional grey difference equation, we construct a novel discrete grey Riccati model (DGRM) and analyze its adaptability and properties. Finally, several cases are utilized to evaluate the new model's effectiveness and practicability.

The rest of this investigation is arranged as follows. In Section 2, a new DGRM is advanced. Then, the adaptability of the new model is analyzed, and two essential properties are presented. In Section 3, two numerical examples and an application case are given to check the new model's prediction effect. In Section 4, the conclusions of the investigation are drawn.

2. Discrete grey Riccati model

2.1 The construction of the discrete grey Riccati model

From Wang *et al.* (2019), the general form of the Riccati difference equation can be expressed as:

$$\omega(z+1) = \frac{a(z)\omega(z)+b(z)}{c(z)\omega(z)+d(z)}, \tag{1}$$

in which $a(z)$, $b(z)$, $c(z)$ and $d(z)$ represent rational functions with $a(z)c(z)\neq 0$ and $a(z)d(z)-b(z)c(z)\neq 0$. When $a(z)$, $b(z)$, $c(z)$ and $d(z)$ are constant functions, Eq. (1) can be simplified to the following form:

$$\omega(z+1) = \frac{\alpha\omega(z)+\beta}{\omega(z)+\gamma}, \tag{2}$$

where α, β and γ are real numbers with $\alpha\neq 0$ and $\beta\neq\alpha\gamma$.

Eq. (2) is a special kind of Riccati difference equation. Now, we combine it with a DGM and then construct a new DGRM.

Suppose that $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$ is a non-negative data sequence of a system with n entries, the sequence $X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n))$ is the first-order accumulative generation sequence, where:

$$x^{(1)}(k) = \sum_{j=1}^{k} x^{(0)}(j), \quad k = 1, 2, \cdots, n$$

Definition 1. The equation

$$x^{(1)}(k+1) = \frac{ax^{(1)}(k)+b}{x^{(1)}(k)+c} \quad (a\neq 0, b\neq ac) \tag{3}$$

is called a DGRM, and it is denoted as DGRM(1,1).

From Eq. (3), we have:

$$x^{(1)}(k+1)\left[x^{(1)}(k)+c\right] = ax^{(1)}(k)+b,$$

that is,

$$ax^{(1)}(k)+b-cx^{(1)}(k+1) = x^{(1)}(k)x^{(1)}(k+1)$$

For a given original sequence $X^{(0)}$, the unconstrained optimization problem under the least square criterion of DGRM(1,1) is as follows:

$$\min_{a,b,c} \sum_{k=1}^{n-1} \left[x^{(1)}(k)x^{(1)}(k+1) - ax^{(1)}(k) - b + cx^{(1)}(k+1)\right]^2$$

Thus, the parameters of DGRM(1,1) are estimated by:

$$\widehat{\kappa} = (\widehat{a}, \widehat{b}, \widehat{c})^T = \left(B^T B\right)^{-1} B^T Y \tag{4}$$

with,

$$B=\begin{pmatrix} x^{(1)}(1) & 1 & -x^{(1)}(2) \\ x^{(1)}(2) & 1 & -x^{(1)}(3) \\ \vdots & \vdots & \vdots \\ x^{(1)}(n-1) & 1 & -x^{(1)}(n) \end{pmatrix},\quad Y=\begin{pmatrix} x^{(1)}(1)x^{(1)}(2) \\ x^{(1)}(2)x^{(1)}(3) \\ \vdots \\ x^{(1)}(n-1)x^{(1)}(n) \end{pmatrix}$$

By substituting the obtained parameter values $\hat{\kappa}=(\hat{a},\hat{b},\hat{c})^T$ into Eq. (3), the fitted sequence $\hat{X}^{(1)}$ of DGRM(1,1) satisfies the following relationship:

$$\hat{x}^{(1)}(k+1)=\frac{\hat{a}\hat{x}^{(1)}(k)+\hat{b}}{\hat{x}^{(1)}(k)+\hat{c}}. \tag{5}$$

To obtain the expression of $\hat{x}^{(1)}(k)$ in Eq. (5), we first deform Eq. (5). Let,

$$\hat{x}^{(1)}(k)+\hat{c}=\frac{y(k+1)}{y(k)}\quad (y(k)\neq 0);$$

then,

$$\hat{x}^{(1)}(k)=\frac{y(k+1)}{y(k)}-\hat{c}. \tag{6}$$

By substituting Eq. (6) into Eq. (5), it follows that:

$$\frac{y(k+2)}{y(k+1)}-\hat{c}=\frac{\hat{a}\left(\frac{y(k+1)}{y(k)}-\hat{c}\right)+\hat{b}}{\frac{y(k+1)}{y(k)}}. \tag{7}$$

By simplifying Eq. (7), we can obtain that $y(k)$ satisfies the following equation:

$$y(k+2)-(\hat{a}+\hat{c})y(k+1)+(\hat{a}\hat{c}-\hat{b})y(k)=0. \tag{8}$$

Eq. (8) is a second-order homogeneous difference equation with constant coefficients, so its characteristic equation is:

$$\lambda^2-(\hat{a}+\hat{c})\lambda+\hat{a}\hat{c}-\hat{b}=0. \tag{9}$$

Let $\Delta=(\hat{a}+\hat{c})^2-4(\hat{a}\hat{c}-\hat{b})=(\hat{a}-\hat{c})^2+4\hat{b}$. It follows from Eq. (9) that the two characteristic roots of Eq. (8) can be expressed as:

$$\lambda_1=\frac{\hat{a}+\hat{c}-\sqrt{\Delta}}{2},\quad \lambda_2=\frac{\hat{a}+\hat{c}+\sqrt{\Delta}}{2}. \tag{10}$$

Thus, the general solution of Eq. (8) can be written as:

$$y(k)=\begin{cases} m_1\left(\frac{\hat{a}+\hat{c}-\sqrt{\Delta}}{2}\right)^k+m_2\left(\frac{\hat{a}+\hat{c}+\sqrt{\Delta}}{2}\right)^k, & \Delta>0 \\ (m_3+m_4k)\left(\frac{\hat{a}+\hat{c}}{2}\right)^k, & \Delta=0, \\ m_6(\hat{a}\hat{c}-\hat{b})^{\frac{1}{2}k}\cos(\theta k+m_5), & \Delta<0 \end{cases} \tag{11}$$

in which $\theta = \arctan\left(\frac{\sqrt{-\Delta}}{\widehat{a}+\widehat{c}}\right)$ and $m_i(i = 1, 2, \cdots, 6)$ are arbitrary constants that satisfy that m_1 and m_2 are not both zero, m_3 and m_4 are not both zero and $m_6 \neq 0$.

Substituting Eq. (11) into Eq. (6), $\widehat{x}^{(1)}(k)$ can be expressed as

$$\widehat{x}^{(1)}(k) = \begin{cases} \dfrac{m_1\left(\frac{\widehat{a}+\widehat{c}-\sqrt{\Delta}}{2}\right)^{k+1} + m_2\left(\frac{\widehat{a}+\widehat{c}-\sqrt{\Delta}}{2}\right)^{k+1}}{m_1\left(\frac{\widehat{a}+\widehat{c}-\sqrt{\Delta}}{2}\right)^{k} + m_2\left(\frac{\widehat{a}+\widehat{c}-\sqrt{\Delta}}{2}\right)^{k}} - \widehat{c}, & \Delta > 0 \\ \dfrac{(\widehat{a}+\widehat{c})(m_3 + m_4(k+1))}{2(m_3 + m_4 k)} - \widehat{c}, & \Delta = 0. \\ \dfrac{\sqrt{\widehat{a}\widehat{c} - \widehat{b}}\cos(\theta(k+1) + m_5)}{\cos(\theta k + m_5)} - \widehat{c}, & \Delta < 0 \end{cases} \tag{12}$$

When $\Delta = 0$, the system is in a critical and very unstable state, and it is easily affected by the calculation rounding error, measurement error and sequence length. Additionally, when $\Delta < 0$, it can be seen from Eq. (12) that the fitting function contains two trigonometric functions. Because the trigonometric functions are oscillatory and the first-order accumulative generation sequence of a non-negative sequence is an incremental sequence, it is not appropriate to use an oscillatory sequence to fit an incremental sequence. From the above analysis, the model established when $\Delta \leq 0$ is not suitable for simulation and prediction. Therefore, this investigation only discusses the case when $\Delta > 0$, and this case is very common. That is, the time response formula of the DGRM(1,1) for prediction in this investigation is given by

$$\widehat{x}^{(1)}(k) = \frac{m_1\left(\frac{\widehat{a}+\widehat{c}-\sqrt{\Delta}}{2}\right)^{k+1} + m_2\left(\frac{\widehat{a}+\widehat{c}-\sqrt{\Delta}}{2}\right)^{k+1}}{m_1\left(\frac{\widehat{a}+\widehat{c}-\sqrt{\Delta}}{2}\right)^{k} + m_2\left(\frac{\widehat{a}+\widehat{c}-\sqrt{\Delta}}{2}\right)^{k}} - \widehat{c}. \tag{13}$$

By Eq. (10), If $m_2 \neq 0$, then Eq. (13) can be written as:

$$\begin{aligned} \widehat{x}^{(1)}(k) &= \frac{m_1\lambda_1^{k+1} + m_2\lambda_2^{k+1}}{m_1\lambda_1^k + m_2\lambda_2^k} - \widehat{c} \\ &= \frac{\left(m_1\lambda_1\lambda_1^k + m_2\lambda_1\lambda_2^k\right) - m_2\lambda_1\lambda_2^k + m_2\lambda_2\lambda_2^k}{m_1\lambda_1^k + m_2\lambda_2^k} - \widehat{c} \\ &= \frac{m_2(\lambda_2 - \lambda_1)\lambda_2^k}{m_1\lambda_1^k + m_2\lambda_2^k} + \lambda_1 - \widehat{c} = \frac{m_2(\lambda_2 - \lambda_1)}{m_1\left(\frac{\lambda_1}{\lambda_2}\right)^k + m_2} + \lambda_1 - \widehat{c} \\ &= \frac{\lambda_2 - \lambda_1}{\frac{m_1}{m_2}\left(\frac{\lambda_1}{\lambda_2}\right)^k + 1} + \lambda_1 - \widehat{c}. \end{aligned} \tag{14}$$

Let $m = \frac{m_1}{m_2}$; then, Eq. (14) is equivalent to:

$$\widehat{x}^{(1)}(k) = \frac{\lambda_2 - \lambda_1}{m\left(\frac{\lambda_1}{\lambda_2}\right)^k + 1} + \lambda_1 - \widehat{c}. \tag{15}$$

To determine the value of m, we choose the initial condition as:

$$\widehat{x}^{(1)}(1) = x^{(1)}(1). \tag{16}$$

Substituting Eq. (16) into Eq. (15), we can obtain:

$$m = -\frac{\lambda_2(x^{(1)}(1) + \widehat{c} - \lambda_2)}{\lambda_1(x^{(1)}(1) + \widehat{c} - \lambda_1)} \tag{17}$$

Thus, the time response formula of DGRM(1,1) can be expressed as:

$$\widehat{x}^{(1)}(k) = \frac{\lambda_2 - \lambda_1}{m\left(\frac{\lambda_1}{\lambda_2}\right)^k + 1} + \lambda_1 - \widehat{c}, \quad k = 1, 2, \ldots \tag{18}$$

where $\lambda_{1,2} = \frac{\widehat{a} + \widehat{c} \mp \sqrt{(\widehat{a} - \widehat{c})^2 + 4\widehat{b}}}{2}$ and $m = -\frac{\lambda_2(x^{(1)}(1) + \widehat{c} - \lambda_2)}{\lambda_1(x^{(1)}(1) + \widehat{c} - \lambda_1)}$.

From Eq. (18), we can calculate the simulation and prediction values $\widehat{x}^{(1)}(k)(k = 1, 2, \cdots)$ of the sequence $X^{(1)}$. Because,

$$x^{(1)}(k) = \sum_{j=1}^{k-1} x^{(0)}(j) + x^{(0)}(k) = x^{(1)}(k-1) + x^{(0)}(k).$$

the simulation and prediction values $\widehat{x}^{(0)}(k)(k = 1, 2, \cdots)$ of the original sequence $X^{(0)}$ can be obtained by the following recursive formula:

$$\begin{cases} \widehat{x}^{(0)}(1) = \widehat{x}^{(1)}(1) \\ \widehat{x}^{(0)}(k) = \widehat{x}^{(1)}(k) - \widehat{x}^{(1)}(k-1), \quad k = 2, 3, \cdots \end{cases}$$

2.2 The adaptability analysis of the discrete grey Riccati model

According to the time response formula of DGRM(1,1), we define a continuous function as:

$$x^{(1)}(t) = \frac{\lambda_2 - \lambda_1}{m\left(\frac{\lambda_1}{\lambda_2}\right)^t + 1} + \lambda_1 - c,$$

where $\lambda_{1,2} = \frac{a + c \mp \sqrt{(a - c)^2 + 4b}}{2}$ and $m = -\frac{\lambda_2(x^{(1)}(1) + c - \lambda_2)}{\lambda_1(x^{(1)}(1) + c - \lambda_1)}$.

Clearly, we have that $\lambda_1 < \lambda_2$. Next, we will analyze the monotonicity of $x^{(1)}(t)$ according to the values of λ_1, λ_2 and m.

(1) For $0 < \lambda_1 < \lambda_2$, we can obtain that $\lambda_2 - \lambda_1 > 0$ and $0 < \frac{\lambda_1}{\lambda_2} < 1$. When $m > 0$, $m\left(\frac{\lambda_1}{\lambda_2}\right)^t$ is monotonically decreasing, that is, $\frac{\lambda_2 - \lambda_1}{m\left(\frac{\lambda_1}{\lambda_2}\right)^t + 1}$ is monotonically increasing.

Thus, $x^{(1)}(t)$ is also monotonically increasing. When $m < 0$, $m\left(\frac{\lambda_1}{\lambda_2}\right)^t$ is monotonically increasing, that is, $\frac{\lambda_2-\lambda_1}{m\left(\frac{\lambda_1}{\lambda_2}\right)^t+1}$ is monotonically decreasing. Thus, $x^{(1)}(t)$ is also monotonically decreasing.

(2) For $\lambda_1 < 0 < \lambda_2$, we have that $\lambda_2 - \lambda_1 > 0$ and $\frac{\lambda_1}{\lambda_2} < 0$. When t is a positive integer, $\left(\frac{\lambda_1}{\lambda_2}\right)^t$ is oscillatory, so $x^{(1)}(t)$ is an oscillatory function.

(3) For $\lambda_1 < \lambda_2 < 0$, we have that $\lambda_2 - \lambda_1 > 0$ and $\frac{\lambda_1}{\lambda_2} > 1$. When $m > 0$, $m\left(\frac{\lambda_1}{\lambda_2}\right)^t$ is monotonically increasing, that is, $\frac{\lambda_2-\lambda_1}{m\left(\frac{\lambda_1}{\lambda_2}\right)^t+1}$ is monotonically decreasing. Thus, $x^{(1)}(t)$ is also monotonically decreasing. When $m < 0$, $m\left(\frac{\lambda_1}{\lambda_2}\right)^t$ is monotonically decreasing, that is, $\frac{\lambda_2-\lambda_1}{m\left(\frac{\lambda_1}{\lambda_2}\right)^t+1}$ is monotonically increasing. Thus, $x^{(1)}(t)$ is also monotonically increasing.

In summary, we can obtain the change characteristics of function $x^{(1)}(t)$ in various cases and list the detailed results in Table 1.

Since $X^{(1)} = (x^{(1)}(k)|k = 1, 2, \cdots, n)$ is the first-order accumulative generation sequence of a non-negative original sequence, $x^{(1)}(k)$ increases with the increase of k. Therefore, we only analyze the case when the function $x^{(1)}(t)$ is monotonically increasing. Now, we analyze the asymptotic behavior of $x^{(1)}(t)$ when it is increasing.

(1) When $0 < \lambda_1 < \lambda_2$ and $m > 0$, we have $0 < \frac{\lambda_1}{\lambda_2} < 1$. If $t \to +\infty$, then $\left(\frac{\lambda_1}{\lambda_2}\right)^t \to 0$. It follows that:

$$x^{(1)}(t) \to \lambda_2 - c$$

(2) When $\lambda_1 < \lambda_2 < 0$ and $m < 0$, we have $\frac{\lambda_1}{\lambda_2} > 1$. If $t \to +\infty$, then $\left(\frac{\lambda_1}{\lambda_2}\right)^t \to +\infty$. It follows that:

$$x^{(1)}(t) \to \lambda_1 - c.$$

Next, we will analyze the concavity and convexity of $x^{(1)}(t)$ when it is increasing. To simplify the expression of $x^{(1)}(t)$, we let $\lambda = \frac{\lambda_1}{\lambda_2}$ and $\delta = \lambda_2 - \lambda_1$. Then,

$$x^{(1)}(t) = \frac{\delta}{m\lambda^t + 1} + \lambda_1 - c. \tag{19}$$

Therefore, the derivative of $x^{(1)}(t)$ can be calculated as:

$$\frac{\mathrm{d}x^{(1)}(t)}{\mathrm{d}t} = -\frac{\delta}{(m\lambda^t+1)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(m\lambda^t+1) = -\frac{m\delta\ln\lambda}{(m\lambda^t+1)^2}\lambda^t,$$

Serial number	λ_1	λ_2	m	$x^{(1)}(t)$
1	+	+	+	Monotonically increasing
2	+	+	−	Monotonically decreasing
3	−	+	+ or −	Oscillatory
4	−	−	+	Monotonically decreasing
5	−	−	−	Monotonically increasing

Table 1. The change characteristics of the function $x^{(1)}(t)$ in various cases

and the second derivative of $x^{(1)}(t)$ can be calculated as:

$$\begin{aligned}\frac{\mathrm{d}^2x^{(1)}(t)}{\mathrm{d}t^2} &= -m\delta\ln\lambda\cdot\frac{d}{dt}\left[\frac{\lambda^t}{(m\lambda^t+1)^2}\right]\\ &= -m\delta\ln\lambda\cdot\frac{\lambda^t\ln\lambda(m\lambda^t+1)^2-\lambda^t\cdot 2(m\lambda^t+1)\cdot m\lambda^t\ln\lambda}{(m\lambda^t+1)^4}\\ &= -\frac{m\delta\ln\lambda}{(m\lambda^t+1)^4}\cdot\left(\ln\lambda\cdot\lambda^t - m^2\ln\lambda\cdot\lambda^{3t}\right)\\ &= -\frac{m\delta(\ln\lambda)^2\lambda^t}{(m\lambda^t+1)^4}\cdot\left(1-m^2\lambda^{2t}\right)\end{aligned}$$

Here, we only consider the general case when $\lambda\neq 1$ and $m\neq 0$. Let $\frac{\mathrm{d}^2x^{(1)}(t)}{\mathrm{d}t^2}=0$, then $1-m^2\lambda^{2t}=0$. Thus, we can get a possible inflection point $t_0=-\log_\lambda^{|m|}$.

(1) If $m>0$ and $0<\lambda<1$, then $\frac{\mathrm{d}^2x^{(1)}(t)}{\mathrm{d}t^2}>0$ for $t<t_0$ and $\frac{\mathrm{d}^2x^{(1)}(t)}{\mathrm{d}t^2}<0$ for $t>t_0$;

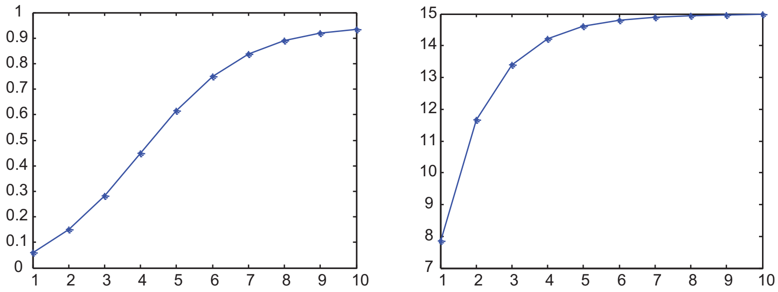
(2) If $m<0$ and $\lambda>1$, then $\frac{\mathrm{d}^2x^{(1)}(t)}{\mathrm{d}t^2}>0$ for $t<t_0$ and $\frac{\mathrm{d}^2x^{(1)}(t)}{\mathrm{d}t^2}<0$ for $t>t_0$.

In the above two cases, $x^{(1)}(t)$ is shown as an increasing curve that changes from concave to convex, which is also called an S-type curve. For time series $x^{(1)}(k)\ \ (k=1,2,\cdots)$, if $t_0>1$, then $x^{(1)}(k)$ is an *S*-type curve, and $x^{(0)}(k)$ is a unimodal curve that changes from increasing to decreasing. If $t_0\leq 1$, then $x^{(1)}(k)$ is a convex and increasing curve, and $x^{(0)}(k)$ is a decreasing curve. Figure 1 depicts the changing curves of $x^{(1)}(k)$ in two different cases.

From the above analysis, DGRM(1,1) is fit for the simulation and prediction of a single peak sequence and a decreasing sequence. In fact, if a single peak sequence is divided into two stages: an increasing stage and a decreasing stage, then a decreasing sequence can be looked upon as the second stage of a single peak sequence. Therefore, DGRM(1,1) is also appropriate for the short-term prediction of the first stage of a single peak sequence (i.e. an incremental sequence).

2.3 Two properties of the discrete grey Riccati model

Proposition 1. Suppose that the estimated parameters of DGRM(1,1) for nonnegative sequences $X^{(0)}=(x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n))$ and $X_d^{(0)}=(x_d^{(0)}(1),x_d^{(0)}(2),\cdots,x_d^{(0)}(n))$ are $\widehat{\kappa}=(\widehat{a},\widehat{b},\widehat{c})^T$ and $\widehat{\kappa}'=(\widehat{a}',\widehat{b}',\widehat{c}')^T$, respectively. If $X_d^{(0)}=\rho X^{(0)}(\rho>0)$, then $\widehat{a}'=\rho\widehat{a}$, $\widehat{b}'=\rho^2\widehat{b}$ and $\widehat{c}'=\rho\widehat{c}$.



(1) $\lambda=5.0$, $m=16$, $\delta=1$, $\lambda_1-c=-0.05$ (2) $\lambda=2$, $m=-4$, $\delta=50$, $\lambda_1-c=15$

Figure 1.
The changing curves of $x^{(1)}(k)$ in two different cases

Proof from Eq. (3), we have:

$$ax^{(1)}(k)+b-cx^{(1)}(k+1)=x^{(1)}(k)x^{(1)}(k+1).$$

Let,

$$B=\begin{pmatrix} x^{(1)}(1) & 1 & -x^{(1)}(2) \\ x^{(1)}(2) & 1 & -x^{(1)}(3) \\ \vdots & \vdots & \vdots \\ x^{(1)}(n-1) & 1 & -x^{(1)}(n) \end{pmatrix} \text{ and } Y=\begin{pmatrix} x^{(1)}(1)x^{(1)}(2) \\ x^{(1)}(2)x^{(1)}(3) \\ \vdots \\ x^{(1)}(n-1)x^{(1)}(n) \end{pmatrix}$$

Then, the parameter estimation formula after establishing the DGRM(1,1) model for $X^{(0)}$ is:

$$(\widehat{a},\widehat{b},\widehat{c})^T=(B^TB)^{-1}B^TY.$$

Since $X_d^{(0)}=\rho X^{(0)}$, $X_d^{(1)}=(x_d^{(1)}(1),x_d^{(1)}(2),\cdots,x_d^{(1)}(n))$, which is the first-order accumulative generation sequence of $X_d^{(0)}$, satisfies:

$$x_d^{(1)}(k)=\sum_{j=1}^{k}x_d^{(0)}(j)=\sum_{j=1}^{k}\rho x^{(0)}(j)=\rho\sum_{j=1}^{k}x^{(0)}(j)=\rho x^{(1)}(k),\quad k=1,2,\cdots,n$$

that is, $X_d^{(1)}=\rho X^{(1)}$.

Let,

$$B_1=\begin{pmatrix} x_d^{(1)}(1) & 1 & -x_d^{(1)}(2) \\ x_d^{(1)}(2) & 1 & -x_d^{(1)}(3) \\ \vdots & \vdots & \vdots \\ x_d^{(1)}(n-1) & 1 & -x_d^{(1)}(n) \end{pmatrix} \text{ and } Y_1=\begin{pmatrix} x_d^{(1)}(1)x_d^{(1)}(2) \\ x_d^{(1)}(2)x_d^{(1)}(3) \\ \vdots \\ x_d^{(1)}(n-1)x_d^{(1)}(n) \end{pmatrix}$$

Then, the parameter estimation formula after establishing the DGRM(1,1) model for $X_d^{(0)}$ is:

$$(\widehat{a'},\widehat{b'},\widehat{c'})^T=(B_1^TB_1)^{-1}B_1^TY_1$$

We can obtain that,

$$Y_1=\begin{pmatrix} x_d^{(1)}(1)x_d^{(1)}(2) \\ x_d^{(1)}(2)x_d^{(1)}(3) \\ \vdots \\ x_d^{(1)}(n-1)x_d^{(1)}(n) \end{pmatrix}=\begin{pmatrix} \rho x^{(1)}(1)\cdot\rho x^{(1)}(2) \\ \rho x^{(1)}(2)\cdot\rho x^{(1)}(3) \\ \vdots \\ \rho x^{(1)}(n-1)\cdot\rho x^{(1)}(n) \end{pmatrix}=\rho^2\begin{pmatrix} x^{(1)}(1)x^{(1)}(2) \\ x^{(1)}(2)x^{(1)}(3) \\ \vdots \\ x^{(1)}(n-1)x^{(1)}(n) \end{pmatrix}$$

$$=\rho^2Y,$$

and,

$$B_1 = \begin{pmatrix} x_d^{(1)}(1) & 1 & -x_d^{(1)}(2) \\ x_d^{(1)}(2) & 1 & -x_d^{(1)}(3) \\ \vdots & \vdots & \vdots \\ x_d^{(1)}(n-1) & 1 & -x_d^{(1)}(n) \end{pmatrix} = \begin{pmatrix} \rho x^{(1)}(1) & 1 & -\rho x^{(1)}(2) \\ \rho x^{(1)}(2) & 1 & -\rho x^{(1)}(3) \\ \vdots & \vdots & \vdots \\ \rho x^{(1)}(n-1) & 1 & -\rho x^{(1)}(n) \end{pmatrix} = BM,$$

where M is a non-singular matrix defined as $M = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix}$. Since,

$$M = M^T = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \text{ and } M^{-1} = \begin{pmatrix} \frac{1}{\rho} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\rho} \end{pmatrix},$$

We have,

$$\begin{aligned} (\widehat{a'}, \widehat{b'}, \widehat{c'})^T &= \left(B_1^T B_1\right)^{-1} B_1^T Y_1 = \left[(BM)^T (BM)\right]^{-1} (BM)^T \left(\rho^2 Y\right) \\ &= \rho^2 \left(M^T B^T BM\right)^{-1} M^T B^T Y = \rho^2 M^{-1} \left(B^T B\right)^{-1} \left(M^T\right)^{-1} M^T B^T Y \\ &= \rho^2 M^{-1} \left(B^T B\right)^{-1} B^T Y = \rho^2 \begin{pmatrix} \frac{1}{\rho} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\rho} \end{pmatrix} \left(B^T B\right)^{-1} B^T Y \\ &= \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{pmatrix} = \left(\rho\widehat{a}, \rho^2\widehat{b}, \rho\widehat{c}\right)^T. \end{aligned}$$

Therefore, we obtain that $\widehat{a'} = \rho\widehat{a}$, $\widehat{b'} = \rho^2\widehat{b}$ and $\widehat{c'} = \rho\widehat{c}$.

Proposition 2. Suppose that the fitted sequences of DGRM(1,1) for nonnegative sequences $X^{(0)}$ and $X_d^{(0)}$ are $\widehat{X}^{(0)}$ and $\widehat{X}_d^{(0)}$, respectively. Let,

$$\varepsilon = \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\widehat{x}^{(0)}(k) - x^{(0)}(k)}{x^{(0)}(k)} \right| \text{ and } \varepsilon_d = \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\widehat{x}_d^{(0)}(k) - x_d^{(0)}(k)}{x_d^{(0)}(k)} \right|$$

If $X_d^{(0)} = \rho X^{(0)}$, then $\varepsilon = \varepsilon_d$.

Proof Suppose that the estimated parameters of DGRM(1,1) for nonnegative sequences $X^{(0)}$ and $X_d^{(0)}$ are $\widehat{\kappa} = (\widehat{a}, \widehat{b}, \widehat{c})^T$ and $\widehat{\kappa'} = (\widehat{a'}, \widehat{b'}, \widehat{c'})^T$, respectively. According to Section 2.1, the time response sequence of DGRM(1,1) for $X^{(0)}$ can be expressed as:

$$\widehat{x}^{(1)}(k) = \frac{\lambda_2 - \lambda_1}{m\left(\frac{\lambda_1}{\lambda_2}\right)^k + 1} + \lambda_1 - \widehat{c}, \quad k = 1, 2, \cdots$$

where $\lambda_{1,2} = \frac{\widehat{a}+\widehat{c} \mp \sqrt{(\widehat{a}-\widehat{c})^2 + 4\widehat{b}}}{2}$ and $m = -\frac{\lambda_2(x^{(1)}(1)+\widehat{c}-\lambda_2)}{\lambda_1(x^{(1)}(1)+\widehat{c}-\lambda_1)}$. The time response sequence of DGRM(1,1) for $X_d^{(0)}$ can be expressed as:

$$\widehat{x}_d^{(1)}(k) = \frac{\lambda_2' - \lambda_1'}{m'\left(\frac{\lambda_1'}{\lambda_2'}\right)^k + 1} + \lambda_1' - \widehat{c'}, \quad k = 1, 2, \cdots \tag{20}$$

where $\lambda_{1,2}' = \frac{\widehat{a'}+\widehat{c'} \mp \sqrt{(\widehat{a'}-\widehat{c'})^2 + 4\widehat{b'}}}{2}$ and $m' = -\frac{\lambda_2'(x_d^{(1)}(1)+\widehat{c'}-\lambda_2')}{\lambda_1'(x_d^{(1)}(1)+\widehat{c'}-\lambda_1')}$.

Since $X_d^{(0)} = \rho X^{(0)}$, we have $X_d^{(1)} = \rho X^{(1)}$. Owing to Proposition 1, we have that $\widehat{a'} = \rho\widehat{a}$, $\widehat{b'} = \rho^2\widehat{b}$ and $\widehat{c'} = \rho\widehat{c}$. Therefore, we obtain the following relationship:

$$\lambda_1' = \rho\lambda_1, \ \ \lambda_2' = \rho\lambda_2 \ \text{ and } \ m' = m. \tag{21}$$

By substituting Eq. (21) into Eq. (20), we obtain:

$$\widehat{x}_d^{(1)}(k) = \frac{\rho\lambda_2 - \rho\lambda_1}{m\left(\frac{\rho\lambda_1}{\rho\lambda_2}\right)^k + 1} + \rho\lambda_1 - \rho\widehat{c} = \frac{\rho(\lambda_2 - \lambda_1)}{m\left(\frac{\lambda_1}{\lambda_2}\right)^k + 1} + \rho(\lambda_1 - \widehat{c}) = \rho\widehat{x}^{(1)}(k).$$

Then, the restored values of $x_d^{(0)}(k)$ can be expressed as:

$$\begin{cases} \widehat{x}_d^{(0)}(1) = \widehat{x}_d^{(1)}(1) = \rho\widehat{x}^{(1)}(1) = \rho\widehat{x}^{(0)}(1) \\ \widehat{x}_d^{(0)}(k) = \widehat{x}_d^{(1)}(k) - \widehat{x}_d^{(1)}(k-1) = \rho\widehat{x}^{(1)}(k) - \rho\widehat{x}^{(1)}(k-1) = \rho\widehat{x}^{(0)}(k), \quad k = 2, 3, \cdots \end{cases}$$

It follows that $\widehat{X}_d^{(0)} = \rho\widehat{X}^{(0)}$. Therefore,

$$\varepsilon_d = \frac{1}{n}\sum_{k=1}^{n}\left|\frac{\widehat{x}_d^{(0)}(k) - x_d^{(0)}(k)}{x_d^{(0)}(k)}\right| = \frac{1}{n}\sum_{k=1}^{n}\left|\frac{\rho\widehat{x}^{(0)}(k) - \rho x^{(0)}(k)}{\rho x^{(0)}(k)}\right| = \frac{1}{n}\sum_{k=1}^{n}\left|\frac{\widehat{x}^{(0)}(k) - x^{(0)}(k)}{x^{(0)}(k)}\right| = \varepsilon.$$

Proposition two shows that the average relative simulation error (ARSE) of DGRM(1,1) does not change if the model is built after the original sequence is transformed with a multiple coefficient ρ. Therefore, selecting an appropriate multiple coefficient can reduce the condition number of B^TB in Eq. (4) and improve the model's reliability and applicability (Cui *et al.*, 2016).

Next, we will provide a numerical example to test the impacts of the multiplier coefficient on the condition number of B^TB and the ARSE of DGRM(1,1). Here, the original time series is constructed as

$$X^{(0)} = (0.4, \ \ 0.8, \ \ 1.4, \ \ 2.2, \ \ 3.3, \ \ 4.6, \ \ 6.1),$$

and the multiple coefficient in the interval (0,1] is taken sequentially at every 0.05 units. Thereafter, the DGRM(1,1) models are established based on different multiple coefficients, and the corresponding spectral condition number of B^TB and the average relative simulation error of the model are obtained. The spectral condition number of B^TB can be calculated as:

$$\text{cond}(B^TB)_2 = ||(B^TB)^{-1}||_2 \cdot ||(B^TB)||_2,$$

where $||\cdot||_2$ denotes the L^2 matrix norm.

Figure 2 reveals the spectral condition numbers of B^TB and the ARSEs of DGRM(1,1) under different multiple coefficients. From it, we can see that the ARSE remains unchanged (1.54%) regardless of the multiple coefficient. Moreover, when the multiple coefficient is taken near 0.35, the condition number can reach the minimum. Therefore, this case test can verify that selecting an appropriate multiple coefficient can effectively reduce the condition number and keep the ARSE unchanged.

3. Case study

To verify the validity and practicability of the grey discrete Riccati model proposed in this investigation and highlight its advantages in prediction, two numerical examples and an application case are tested by the new model and common GMs, such as the grey Verhulst, GM with a polynomial term (GMP(1,1,N)) (Luo and Wei, 2017), discrete GM(1,1) model (DGM(1,1)) (Xie and Liu, 2005) and nonhomogeneous discrete GM(1,1) model (NDGM(1,1)) (Xie *et al.*, 2013), are established to compare with the new model.

3.1 Numerical examples

Case 1. Suppose that a function $x_1(k) = 3e^{0.1k} + k^2 + 10$. Let $k = 1, 2, \cdots, 7$; then, we can obtain a sequence.

$$X_1 = (14.3155,\ 17.6642,\ 23.0496,\ 30.4755,\ 39.9462,\ 51.4664,\ 65.0413).$$

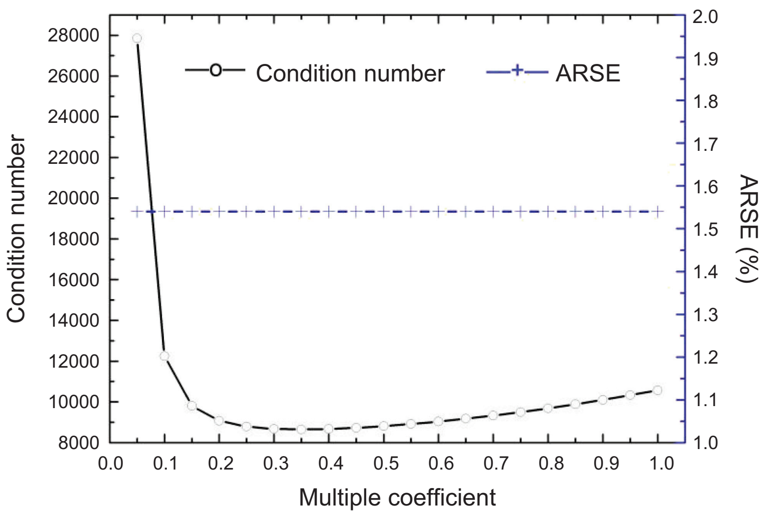


Figure 2.
The spectral condition numbers and the ARSEs under different multiple coefficients

From the data characteristics, the sequence X_1 is a monotonically increasing sequence. Here, we use the first six data points of X_1 to construct the grey Verhulst model, DGM(1,1), GMP(1,1,2), NDGM(1,1) and DGRM(1,1), and the seventh datum is utilized to evaluate the prediction effect of each model. Table 2 reveals their simulation and prediction values and relative errors, and numerics in italic stand for the best results.

From Table 2, the average relative simulation errors of the grey Verhulst model, GPM(1,1,2), DGM(1,1), NDGM(1,1) and DGRM(1,1) are 17.37, 0.004, 0.57, 0.38 and 0.36%, respectively. The results show that DGRM(1,1) is superior to the grey Verhulst model, DGM(1,1) and NDGM(1,1) and inferior to GPM(1,1,2) in simulation performance. In addition, from the perspective of prediction error, the one-step prediction errors of the grey Verhulst model, GPM(1,1,2), DGM(1,1) and NDGM(1,1) are 21.40, 0.01, 3.87 and 2.29%, respectively, while the one-step prediction error of the DGRM(1,1) proposed in this investigation is 0.36%, which is greater than that of GPM(1,1,2) and noticeably smaller than that of the other three models. From the point of view that the simulation and prediction performance of the new model is inferior to that of the GMP(1,1,N), the new model is not the best choice for the simulation and prediction of sequences with a mix of exponential and polynomial characteristics; however, it is worth noting that the number of parameters of the new model is 3, which is less than that of the GPM(1,1,2), indicating that the structure of the new model is relatively simple compared to the structure of the GPM(1,1,2). The above analysis results show that DGRM(1,1) is suitable for the simulation and prediction of a monotonically increasing sequence.

Case 2. Suppose that a function $x_2(k) = \frac{100e^{-0.2k}}{\sqrt{k}} + 2$. Let $k = 1, 2, \cdots, 7$; then, we can obtain a sequence.

$$X_2 = (83.8731, 49.3988, 33.6857, 24.4664, 18.4521, 14.2962, 11.3205).$$

From the data characteristics, the sequence X_2 is a monotonically decreasing sequence. Here, we use the first six data points of X_2 to construct the grey Verhulst model, GPM(1,1,2), DGM(1,1), NDGM(1,1) and DGRM(1,1), and the seventh datum is utilized to evaluate the prediction effect of each model. Table 3 reveals their simulation and prediction values and relative errors.

From Table 3, the average relative simulation errors of the grey Verhulst model, GPM(1,1,2), DGM(1,1) and NDGM(1,1) are 14.11, 0.66%, 3.09% and 0.64%, respectively, while the average simulation relative error of DGRM(1,1) is 0.14%, which is noticeably smaller than that of the former four models. The maximum simulation errors of the five models are 31.22%, 1.57%, 8.38%, 1.27% and 0.29%, respectively, which show that DGRM(1,1) exhibits the best stability. Additionally, from the perspective of prediction error, the one-step prediction errors of the grey Verhulst model, GPM(1,1,2), DGM(1,1) and NDGM(1,1) are 55.30, 2.16, 16.59 and 6.90%, respectively, while the one-step prediction error of DGRM(1,1) proposed in this investigation is 0.97%, which shows that the prediction accuracy of the proposed model is significantly higher than that of the other four models. In conclusion, DGRM(1,1) is fit for the simulation and prediction of a monotonically decreasing sequence.

3.2 Application

In recent years, China's demand for general aircraft has gradually increased with the rapid development of its economy. Because a country's aircraft manufacturing and scientific research capabilities determine the development level of its general aviation industry, the Chinese government has increased its research and development investment in the general aviation industry and actively promoted its rapid development. Accurate prediction of the development cost of a specific type of aircraft is conducive to the rational use and allocation of

Table 2.
Simulation and prediction results of the five models for X_1

		Grey Verhulst		GPM(1,1,2)		DGM(1,1)		NDGM(1,1)		DGRM(1,1)	
Serial number	Real values	Values	Error (%)	Values	Error (%)	Values	Error (%)	Values	Error (%)	Values	Error (%)
1	14.3155	14.3155	0.00	14.3155	0.00	14.3155	0.00	14.3155	0.00	14.3155	0.00
2	17.6642	10.2068	42.22	17.6634	0.00	17.8108	0.83	17.5362	0.72	17.4641	1.13
3	23.0496	16.6251	27.87	23.0484	0.01	23.2529	0.88	23.2446	0.85	23.1801	0.57
4	30.4755	25.6704	15.77	30.4739	0.01	30.3580	0.39	30.5112	0.12	30.5477	0.24
5	39.9462	36.5401	8.53	39.9443	0.00	39.6341	0.78	39.7616	0.46	39.8795	0.17
6	51.4664	46.4134	9.82	51.4641	0.00	51.7445	0.54	51.5373	0.14	51.4268	0.08
The average error (%)			17.37		*0.004*		0.57		0.38		0.36
7	65.0413	51.1211	21.40	65.0378	*0.01*	67.5553	3.87	66.5276	2.29	65.2778	0.36

Table 3. Simulation and prediction results of the five models for X_2

Serial number	Real values	Grey Verhulst		GPM(1,1,2)		DGM(1,1)		NDGM(1,1)		DGRM(1,1)	
		Values	Error (%)	Values	Error (%)	Values	Error (%)	Values	Error (%)	Values	Error (%)
1	83.8731	83.8731	0.00	83.8731	0.00	83.8731	0.00	83.8731	0.00	83.8731	0.00
2	49.3988	41.4589	16.07	48.6235	1.57	48.4910	1.84	49.3155	0.17	49.3864	0.03
3	33.6857	39.0598	15.95	33.6211	0.19	34.9578	3.78	33.9470	0.78	33.7382	0.16
4	24.4664	28.9656	18.39	24.5530	0.35	25.2016	3.01	24.3100	0.64	24.4187	0.20
5	18.4521	17.8917	3.04	18.6425	1.03	18.1682	1.54	18.2669	1.00	18.4218	0.16
6	14.2962	9.8333	31.22	14.4122	0.81	13.0977	8.38	14.4776	1.27	14.3371	0.29
The average error (%)			14.11		0.66		3.09		0.64		*0.14*
7	11.3205	5.0599	55.30	11.0759	2.16	9.4423	16.59	12.1014	6.90	11.4307	*0.97*

Table 4.
Simulation and prediction results of the five models for the development cost of a small plane

Years	Real values	Grey Verhulst		GPM(1,1,3)		DGM(1,1)		NDGM(1,1)		DGRM(1,1)	
		Values	Error (%)	Values	Error (%)	Values	Error (%)	Values	Error (%)	Values	Error (%)
2006	500	500.0	0.00	500.0	0.00	500.0	0.00	500.0	0.00	500.0	0.00
2007	770	736.2	4.39	791.8	2.82	1151.1	49.50	1124.8	46.08	774.5	0.59
2008	1220	1156.4	5.22	1126.9	0.56	888.1	27.21	969.6	20.52	1212.5	0.62
2009	1060	1060.3	0.03	1012.2	4.50	685.1	35.36	791.7	25.31	1063.0	0.29
2010	545	587.8	7.86	570.3	4.64	528.6	3.02	587.8	7.85	551.9	1.27
2011	219	239.2	9.22	193.3	11.75	407.8	86.20	353.9	61.60	211.1	3.61
2012	72	84.8	17.80	83.6	16.05	314.6	336.93	85.7	19.09	71.0	1.37
The average error (%)			6.36		5.76		76.89		25.78		*1.11*
2013	23	28.6	24.26	381.2	1557.20	242.7	955.21	−221.7	1063.94	22.8	*0.69*

investment funds by airlines. In this investigation, we take the use of the research and development cost of a small plane owned by the Aviation Industry Corporation of China (AVIC) as an example and select the data from 2006 to 2013 (500, 770, 1,220, 1,060, 545, 219, 72, 23, unit: ten thousand yuan) (Ding *et al.*, 2015) as an original sequence. From the changing characteristics of the data, the sequence first shows an increasing trend and then shows a decreasing trend, which is a single peak data sequence. Here, we employ the data from 2006 to 2012 to establish the grey Verhulst model, GPM(1,1,3), DGM(1,1), NDGM(1,1) and DGRM(1,1), respectively, and the data from 2013 are utilized to evaluate the prediction effect of each model. Table 4 reveals their simulation and prediction values and relative errors.

From Table 4, the average relative simulation errors of the grey Verhulst model, GPM(1,1,3), DGM(1,1) and NDGM(1,1) are 6.36, 5.76, 76.89 and 25.78%, respectively, while the average simulation relative errors of DGRM(1,1) proposed in this investigation are 1.11%, which is noticeably smaller than that of the former four models. The maximum simulation error of DGRM(1,1) is 3.61%, which is the smallest of the five competing models, and it indicates that DGRM(1,1) exhibits the best stability. Additionally, from the one-step prediction error and prediction value, the prediction errors of the grey Verhulst model, GPM(1,1,3), DGM(1,1) and NDGM(1,1) are 24.26, 1557.20, 955.21 and 1063.94%, respectively, and the prediction value of NDGM(1,1) is negative. Obviously, GPM(1,1,3), DGM(1,1) and NDGM(1,1) are not appropriate to simulate and predict the development cost. The one-step prediction error of DGRM(1,1) is only 0.69%, which is noticeably smaller than that of the other four models. In conclusion, DGRM(1,1) can precisely reflect the changing trend of the development cost of a small plane, and it shows that DGRM(1,1) is suitable for the simulation and prediction of a single peak sequence, which exhibits good application potential.

4. Conclusions

In this investigation, the Riccati difference equation and GM are combined to construct a novel DGRM, and it is theoretically found that the new model is appropriate to predict monotonic increasing, monotonic decreasing and unimodal sequences. Finally, two numerical examples and an application case are used for testing, and the outcomes indicate that the new model is effective and practical compared with the other conventional GMs, confirming the theoretical analysis results. The new model is a supplement to the grey prediction model, which develops the theory and application of the grey prediction model. Of course, there are still some problems worthy of further study, such as the stability and further optimization of the new model.

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Corresponding author
Liang Zeng can be contacted at: zengliang19820809@126.com